Probabilistic Analysis of Foundation Settlement

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Abstract

It is at least intuitively evident that variability in soil properties will have a significant effect on total and differential settlement of structural foundations. By modeling soils as spatially random media, whose properties follow certain distributions and spatial correlation structures, estimates of the reliability of foundations against serviceability limit state failure, in the form of excessive differential settlements, can in principle be made. The soil's property of interest is it's elastic modulus, E, which is represented here using a lognormal marginal distribution and an isotropic correlation structure. Prediction of settlement below a foundation can then be made using the finite element method given a realization of the elastic modulus field underlying the foundation. By generating and analyzing multiple realizations, the statistics and density functions of total and differential settlements can be estimated.

This paper estimates probabilistic measures of total settlement under a single spread footing and of differential settlement under a pair of spread footings using a two-dimensional model combined with a Monte Carlo simulation. For the cases considered, total settlement is found to be well represented by a lognormal distribution and simple relationships are proposed allowing the approximation of the settlement distribution parameters for a footing founded on a spatially random soil of constant depth and fixed Poisson's ratio. A one-parameter exponential distribution is fitted to differential settlements and found to give reasonable probability estimates, particularly towards the tail of the distribution. A method of predicting the single parameter is given in terms of statistics of the elastic modulus field and local averages over the field. An example is presented to illustrate the proposed methodology for a single footing.

INTRODUCTION

The settlement of structures founded on soil is a subject of considerable interest to practicing engineers since excessive settlements often lead to serviceability problems. In particular, unless the total settlements themselves are particularly large, it is

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actually differential settlements which lead to unsightly cracks in facades and structural elements, possibly even to structural failure (especially in unreinforced masonry elements). Existing code requirements limiting differential settlements to satisfy serviceability limit states (see building codes ACI 318-89, 1989, or A23.3-M84, 1984) specify maximum deflections ranging from $D/180$ to $D/480$, depending on the type of supported elements, where D is the center-to-center span of the structural element. In practice, differential settlements between footings are generally controlled, not by considering the differential settlement itself, but by controlling the total settlement predicted by analysis using an estimate of the soil elasticity. This approach is largely based on correlations between total settlements and differential settlements observed experimentally (see for example D'Appolonia *et.al.*,1968) and leads to a limitation of 4 to 8 cm in total settlement under a footing as stipulated by the Canadian Foundation Engineering Manual, Part 2 (1978).

Because of the wide variety of soil types and possible loading conditions, experimental data on differential settlement of footings founded on soil is limited. With the aid of modern high-speed computers, it is now possible to probabilistically investigate differential settlements over a range of loading conditions and geometries. This paper reports the initial findings of such a study and attempts to provide a relatively simple, albeit approximate, approach to estimating probabilities associated with settlements. The paper first considers the case of a single footing, as shown in Figure 1(a), and estimates the probability density function (PDF) governing total settlement of the footing as a function of footing width for various input statistics of the underlying soil. All other parameters are held constant. The footing is assumed to be founded on a soil layer underlain by bedrock. The results are generalized to allow the estimation of probabilities associated with total settlement under an isolated footing in many practical cases. It is emphasized, however, that the results are still preliminary, there being still many aspects of the problem that need investigation. Thus, the results presented in this paper should be viewed as providing only ball-park estimates in the absence of further theoretical and/or empirical developments.

The second part of the paper addresses the issue of differential settlements under a pair of footings, as shown in Figure 1(b), again for the particular case of footings founded on a soil layer underlain by bedrock. The mean and standard deviation of differential settlements are estimated as a function of footing width for various input statistics of the underlying elastic modulus field. Unfortunately, the probability density function governing differential settlement is as yet unknown and only rough estimates of probabilities associated with differential settlement can be made (barring numerical integration of a joint probability density function). In this paper a simple one-parameter exponential distribution is fitted to the simulation data. Since such a simple distribution cannot hope to capture the intricacies of the actual distribution, the fit is aimed at yielding reasonably accurate probability estimates in the tail of the distribution for the particular geometry shown in Figure 1(b).

The physical problem is represented using a two-dimensional model. If the footings extend for a large distance in the out-of-plane direction, z, then the 2-D elastic modulus field is interpreted either as an average over z or as having an infinite scale of fluctuation in the z direction. For footings of finite dimension, the 2-D model is admittedly just an approximation. However, the approximation would be reasonable if the elastic modulus were suitably averaged in the z direction. These issues are not addressed in here and thus the derived 2-D results must be viewed with caution pending a 3-D sensitivity study.

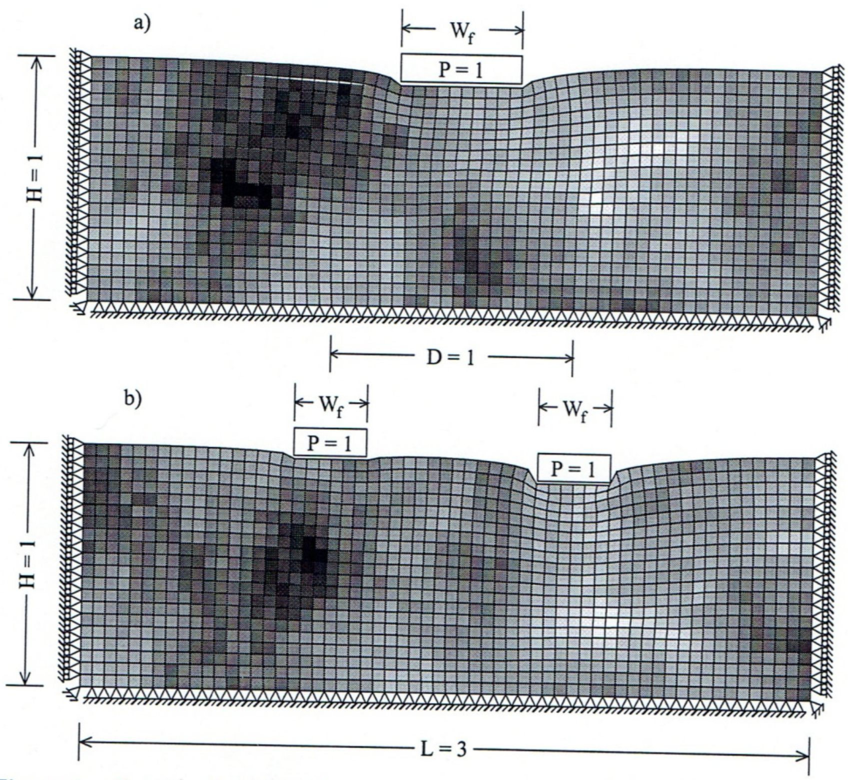

Figure 1. Random field/FEM representation of a) a single footing, and b) two footings founded on a soil layer.

THE RANDOM FIELD/FEM MODEL

As illustrated in Figure 1, the soil mass is discretized into 60 four-noded quadrilateral elements in the horizontal direction by 20 elements in the vertical direction. The overall dimensions of the soil model are held fixed at $L = 3$ in width by $H = 1$ in height. Herein, parameters will be expressed without units, it being understood that a consistent set of units are to be used throughout. The left and right faces of the finite element model are constrained against horizontal displacement but are free to slide vertically while the nodes on the bottom boundary are spatially fixed. The footing(s) are assumed to be rigid, to not undergo any rotations, and to have a rough interface with the underlying soil (no-slip boundary).

To investigate the effect of the ratio of footing width to soil layer thickness, W_f/H, H was held constant at 1.0 while the footing width was varied according to Table 1. In the two footing case, the distance between footing centers was held constant at 1.0, while the footing widths (assumed equal) were varied. In the latter case, footings of width greater than 0.5 were not considered since this situation approaches that of a strip footing (the footings would be joined when $W_f = 1.0$). In all cases, the footing loads P were held constant at 1.0.

The soil has two properties of interest to the settlement problem: these are the elastic modulus, $E(\underset{\sim}{x})$, and Poisson's ratio, $\nu(\underset{\sim}{x})$, where $\underset{\sim}{x}$ is spatial position. At this time for simplicity, only the elastic modulus is considered to be a spatially random property, it being felt that it is the more important variable as far as settlement is concerned. Poisson's ratio is held fixed at 0.25 for all analyses over the entire soil mass. The extension of the results to spatially random Poisson's ratio is reserved for future work.

Table 1. Input parameters varied in the study while holding $H = 1$, $D = 1$, $P = 1$, $\mu_E = 1$, and $\nu = 0.25$ constant.

Parameter	Values Considered
σ_E	0.1, 0.5, 1.0, 2.0, 4.0
$\theta_{\ln E}$	0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 1.0, 2.0, 5.0, 10.0
W_f	0.1, 0.2, 0.5, 1.0 (single footing)
	0.1, 0.3, 0.5 (two footings)

Figure 1 shows, along with the finite element mesh, a grey-scale representation of a possible realization of the elastic modulus field. Lighter areas denote smaller values of $E(\underset{\sim}{x})$ so that the elastic modulus field shown in Figure 1(b) corresponds to a higher elastic modulus under the left footing than under the right – this leads to the substantial differential settlement indicated by the deformed mesh. This is just one possible realization of the E field; the next realization could just as easily show the opposite trend, or perhaps something in between.

The elastic modulus field is assumed to follow a lognormal distribution so that $\ln(E)$ is a Gaussian (normal) random field with mean $\mu_{\ln E}$, and variance $\sigma^2_{\ln E}$. The choice of a lognormal distribution is motivated by the fact that the elastic modulus is strictly positive, as stipulated by the lognormal distribution, while having a simple relationship with the normal distribution. Note that the normal distribution admits negative values of E with non-zero probability. The spatial dependence is assumed to follow an isotropic Gauss-Markov correlation function

$$\rho_{\ln E}(\underset{\sim}{\tau}) = \exp\left\{-\frac{2|\underset{\sim}{\tau}|}{\theta_{\ln E}}\right\} \tag{1}$$

in which $\underset{\sim}{\tau} = \underset{\sim}{x} - \underset{\sim}{x}'$ is the vector between spatial points $\underset{\sim}{x}$ and $\underset{\sim}{x}'$, and $|\underset{\sim}{\tau}|$ is the absolute length of this vector (the lag distance). In this paper, the word 'correlation' refers to the correlation coefficient (normalized covariance). The correlation function decay rate is governed by the so-called scale of fluctuation, $\theta_{\ln E}$, which, loosely speaking, is the distance over which elastic moduli are significantly correlated (when the separation distance $|\underset{\sim}{\tau}|$ is greater than $\theta_{\ln E}$, the correlation between $E(\underset{\sim}{x})$ and $E(\underset{\sim}{x}')$ is less than 14%).

The assumption of isotropy is, admittedly, somewhat restrictive. Although an ellipsoidally anisotropic random field can be converted to an isotropic random field by suitably stretching the coordinate axes, this transformation cannot be performed in a settlement study since the stress field needs to be preserved. In principal the methodology presented in the following is easily extended to anisotropic fields, however, the accuracy of the proposed distribution parameter estimates would need to be verified. In the meantime, the isotropic case is selected for simplicity.

In practice, one approach to the estimation of $\theta_{\ln E}$ involves collecting elastic modulus data from a series of locations in space, estimating the correlations between the log-data as a function of separation distance, and then fitting Eq. (1) to the estimated correlations. See, e.g., Degroot and Baecher (1993), de Marsily (July 1985), Asaoka and Grivas (May 1982), Ravi (1992), Soulié *et.al.*(1990),and Chiasson *et.al.*(1995)for further information on the characterization of spatial variability of soil properties.

Throughout, the mean elastic modulus, μ_E, is held fixed at 1.0. Since settlement varies linearly with the soil elastic modulus, it is always possible to scale the settlement statistics to the actual mean elastic modulus. The standard deviation of the elastic modulus is varied from 0.1 to 4.0 to investigate the effects of elastic modulus variability on settlement variability. The parameters of the transformed $\ln(E)$ Gaussian random field may be obtained from the relations,

$$\sigma^2_{\ln E} = \ln(1 + \sigma^2_E/\mu^2_E) \tag{2a}$$

$$\mu_{\ln E} = \ln(\mu_E) - \tfrac{1}{2}\sigma^2_{\ln E} \tag{2b}$$

from which it can be seen that the variance of $\ln(E)$, $\sigma^2_{\ln E}$, varies from 0.01 to 2.83 (note that the mean of $\ln(E)$ is not constant).

To investigate the effect of the scale of fluctuation, $\theta_{\ln E}$, on the settlement statistics, $\theta_{\ln E}$ is varied from 0.01 (i.e., very much smaller than the soil model size) to 10.0 (i.e., substantially bigger than the soil model size). In the limit as $\theta_{\ln E} \rightarrow 0$, the elastic modulus field becomes a white noise field, with E values at any two distinct points independent. In terms of the finite elements themselves, values of $\theta_{\ln E}$ smaller than the elements results in a set of elements which are largely independent (increasingly independent as $\theta_{\ln E}$ decreases). Because of the averaging effect of the details of the elastic modulus field under a footing, the settlement in the limiting case $\theta_{\ln E} \rightarrow 0$ is expected to approach that obtained in the deterministic case, with $E = \mu_E$ everywhere, and has vanishing variance. By similar reasoning, the differential settlement in this case (as in Figure 1b) is expected to go to zero. At the other extreme, as $\theta_{\ln E} \rightarrow \infty$, the elastic modulus field becomes the same everywhere (different from realization to realization, according to the lognormal distribution, but spatially constant within any one realization). In this case, the settlement statistics are expected to approach those obtained by using a single lognormally distributed random variable, E, to model the soil, $E(\underset{\sim}{x}) = E$. That is, if the settlement, δ, under a footing founded on a soil layer with uniform (but random) elastic modulus E is given by $\delta = \delta_{det}\mu_E/E$, for δ_{det} the settlement when $E = \mu_E$ everywhere, then as $\theta_{\ln E} \rightarrow \infty$ the settlement assumes a lognormal distribution with parameters

$$\mu_{\ln \delta} = \ln(\delta_{det}) + \ln(\mu_E) - \mu_{\ln E} = \ln(\delta_{det}) + \tfrac{1}{2}\sigma^2_{\ln E} \tag{3a}$$

$$\sigma_{\ln \delta} = \sigma_{\ln E} \tag{3b}$$

where Eq. (2b) was used in Eq. (3a). Also since, in this case, the settlement under the two footings of Figure 1(b) becomes equal, the differential settlement becomes zero. Thus, the differential settlement is expected to approach zero both at very small and at very large scales of fluctuation.

Because the variability of the elastic modulus field to be considered can be quite large, up to a COV $= \sigma_E/\mu_E = 4$, and because, perhaps more importantly, it is desired to estimate the entire probability density function (PDF) of settlement, the approach taken herein is via Monte Carlo simulations. Traditional stochastic finite element techniques, involving a first or second order perturbation of the random parameters, cannot be used since they are inaccurate for COV's in excess of about 20% and since they do not provide an estimate of the entire PDF. The Monte Carlo approach adopted here involves the simulation of a realization of the elastic modulus field and subsequent finite element analysis of that realization to yield a realization of the footing settlement(s). Repeating the process over an ensemble of realizations generates a set of possible settlements which can be plotted in the form of a histogram and from which distribution parameters can be estimated. In this study, 2000 realizations are performed for each input parameter set (σ_E, $\theta_{\ln E}$, and W_f). If it can be assumed that log-settlement is approximately normally distributed (which is seen later to be a reasonable assumption and is consistent with the distribution selected for E), and $m_{\ln\delta}$ and $s^2_{\ln\delta}$ are the estimators of the mean and variance of log-settlement, respectively, then the standard deviation of these estimors obtained from 2000 realizations are given by $\sigma_{m_{\ln\delta}} \simeq s_{\ln\delta}/\sqrt{n} = 0.022 s_{\ln\delta}$ and $\sigma_{s^2_{\ln\delta}} \simeq \sqrt{\frac{2}{n-1}}\, s^2_{\ln\delta} = 0.032 s^2_{\ln\delta}$ so that the estimator 'error' is negligible compared to the estimated variance.

Realizations of the log-elastic modulus field, $G(\underset{\sim}{x}_i)$, are produced using the two-dimensional Local Average Subdivision (LAS) technique (Fenton and Vanmarcke, 1990, Fenton, 1994), where $G(\underset{\sim}{x}_i)$ is the local average of a zero mean, unit variance Gaussian random field over the domain of the element centered at $\underset{\sim}{x}_i$. The generated field correctly reproduces the mean, variance and covariance structure of the 2-D local average process. The elastic modulus value then assigned to the i'th element is

$$E(\underset{\sim}{x}_i) = \exp\{\mu_{\ln E} + \sigma_{\ln E}\, G(\underset{\sim}{x}_i)\} \tag{4}$$

Once the field of elastic modulus values is assigned, the settlement(s) are computed via finite element analysis.

SINGLE FOOTING CASE

A typical histogram of the settlement under a single footing, as estimated by 2000 realizations, is shown in Figure 2. This is for the case where the footing has width $W_f/H = 0.2$, $\sigma_E/\mu_E = 2$, and $\theta_{\ln E} = 0.7$. With the requirement that settlement be non-negative, the shape of the histogram suggests a lognormal distribution, which was adopted in this study (see also Eq. 3) . The histogram itself is computed over 30 equally spaced intervals between $\ln(x_{min})$ and $\ln(x_{max})$, in log-space, where x_{min} and x_{max} are the minimum and maximum settlements observed in the sample of 2000. The histogram is normalized to enclose a unit area and a straight line is drawn between the interval midpoints.

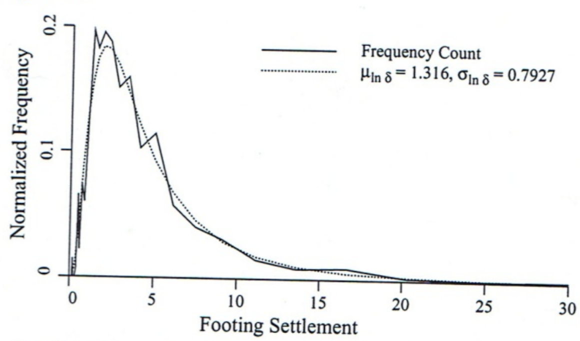

Figure 2. Typical frequency histogram and fitted lognormal distribution of settlement under a single footing.

Superimposed on the histogram is the fitted lognormal distribution with parameters given by $\mu_{\ln\delta}$ and $\sigma_{\ln\delta}$ in the line key. At least visually, the fit appears quite reasonable. In fact a Chi-Square goodness-of-fit test gives a critical p-value of 1×10^{-8}. The critical p-value may be interpreted as the probability of *mistakenly* rejecting the lognormal hypothesis – larger values of p imply a better fit to the data. Unfortunately, the Chi-Square test is quite sensitive to the 'smoothness' of the histogram. Although it would probably be well worth investigating the Kolmogorov-Smirnov goodness-of-fit test to evaluate the fit of the assumed distributions, this was not performed in the current study because the parameters of the assumed distribution are derived from the data and the critical statistic is, strictly speaking, unknown under these conditions.

Over the entire set of simulations done for each parameter set of interest (W_f, σ_E, and $\theta_{\ln E}$) the fraction of critical p-values obtained are listed in Table 2. Since over 30% have critical p-values in excess of 0.05, and over 70% in excess of 0.0001 (and so are better fits than that shown in Figure 2) it appears that the lognormal hypothesis is a reasonable one.

Table 2. Fraction of simulation runs with Chi-Square goodness-of-fit critical p-value greater than that indicated.

p_{crit}	Fraction
> 0.5	7%
> 0.1	26%
> 0.05	33%
> 0.01	49%
> 0.0001	71%

Accepting the lognormal hypothesis as a reasonable fit to the simulation results, the next task is to estimate the parameters of the fitted lognormal distributions as functions of the input parameters (W_f, σ_E, and $\theta_{\ln E}$). The lognormal distribution,

$$f_\delta(x) = \frac{1}{\sqrt{2\pi}\sigma_{\ln\delta}\, x} \exp\left\{-\frac{1}{2}\left(\frac{\ln x - \mu_{\ln\delta}}{\sigma_{\ln\delta}}\right)^2\right\}, \qquad 0 \le x < \infty \tag{5}$$

has two parameters, $\mu_{\ln\delta}$ and $\sigma_{\ln\delta}$. Figure 3 shows how the estimator of $\mu_{\ln\delta}$, $m_{\ln\delta}$, varies with $\sigma_{\ln E}$ for $W_f/H = 0.1$. Similar results were found for the other footing widths. All scales of fluctuation are drawn in the two plots, but are not individually labeled since they lie so close together. This observation implies that the mean log-settlement is largely independent of the scale of fluctuation, $\theta_{\ln E}$. This is as expected since the scale of fluctuation does not affect the mean of a local average of a Gaussian process (recall that if δ is lognormally distributed, then $\ln(\delta)$ is normally distributed). Figure 3 suggests that the mean of log-settlement can be estimated by a straight line of the form

$$\mu_{\ln\delta} = \ln(\delta_{det}) + \alpha_2\,\sigma^2_{\ln E} \tag{6}$$

where δ_{det} is the 'deterministic' settlement obtained from a single finite element analysis (or appropriate approximate calculation) of the problem where $E = \mu_E$ everywhere. For the range of geometries considered in this study, the following relationship can be used to approximate $\ln(\delta_{det})$ reasonably accurately

$$\ln(\delta_{det}) = \ln(P/\mu_E) - 0.4924 - 0.6883\,\ln(W_f/H) - 0.0964\Big(\ln(W_f/H)\Big)^2 \tag{7}$$

which was obtained by regression over the intercepts shown on Figure 3 and over the other footing widths not shown. The r^2 coefficient of determination for the above regression was 0.9999

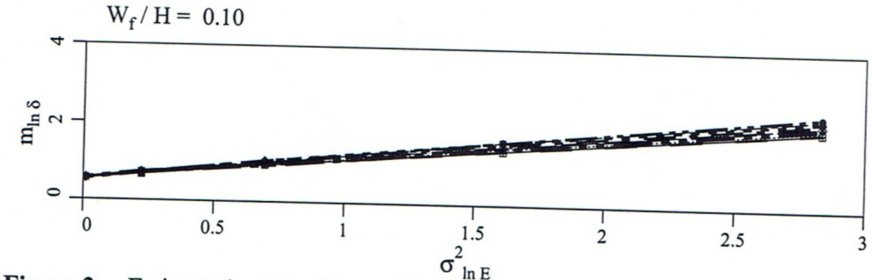

Figure 3. Estimated mean of log-settlement.

The slopes of the curves in Figure 3 are almost uniformly 0.5, as predicted for the settlement by Eq. (3a) in the large and small scale of fluctuation cases. Note that if the settlement mean is independent of the scale of fluctuation, Eq. (3a) is valid for any scale. However, there is in fact a slight dependence of the slope, α_2, on $\theta_{\ln E}$. The second term in the following is a small correction obtained from plots of the slope α_2 versus W_f and $\theta_{\ln E}$

$$\alpha_2 = 0.5 + \frac{0.041}{\sqrt{W_f/H}} \exp\left\{-\tfrac{1}{4}\Big(\ln(\theta_{\ln E}/H)+1\Big)^2\right\} \tag{8}$$

Eq. (8) is entirely empirical, but does have the correct limiting forms for large and small $\theta_{\ln E}$. It is unknown at this time if it can be applied for values of W_f/H outside the range investigated. The physical interpretation and analytical verification of the correction term in the above needs further investigation.

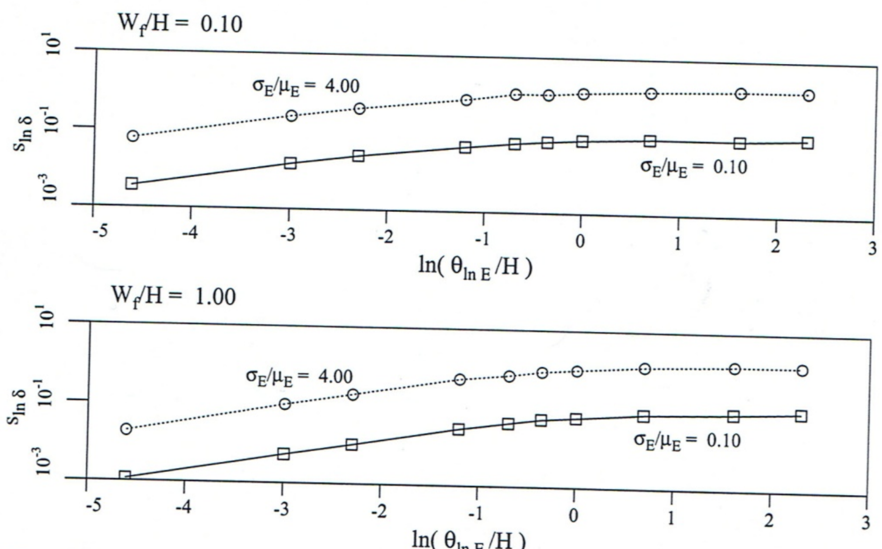

Figure 4. Estimated standard deviation of log-settlement.

The estimator of the standard deviation of log-settlement, $s_{\ln\delta}$, is plotted in Figure 4 for the smallest and largest footing widths. Intermediate footing widths give similar results. In all cases, it can be seen that $s_{\ln\delta} \rightarrow \sigma_{\ln E}$ for large $\theta_{\ln E}$. It is expected that the reduction in variance as $\theta_{\ln E}$ decreases is due largely to the local averaging effect under the footing. That is, if the average of $\ln(E)$ is taken over some area under the footing, then this average is expected to have smaller variance for small scales of fluctuation than for large. This is because there are more 'independent' samples in the area when $\theta_{\ln E}$ is small. Recall that if $\bar{E}$ is the average of samples $E_1, E_2, \ldots, E_n$, then the variance of $\bar{E}$ is σ_E^2/n if the E_i's are mutually independent (the $1/n$ factor is the variance reduction) – this case corresponds to the $\theta_{\ln E} \to 0$ case. On the other hand, if the E_i's are fully correlated ($\theta_{\ln E} \to \infty$), then the variance of $\bar{E}$ is just σ_E^2, so that there is no variance reduction. See Vanmarcke (1984) for more details on local averaging theory. The variance reduction effects are clearly seen in Figure 4.

Following this reasoning, and assuming that local averaging of the area under the footing accounts for all of the variance reduction seen in Figure 4, the standard deviation of log-settlement is

$$\sigma_{\ln \delta} = \sqrt{\gamma(W_f, H;\, \theta_{\ln E})}\, \sigma_{\ln E} \tag{9}$$

in which $\gamma(W_f, H;\, \theta_{\ln E})$, the so-called variance function (Vanmarcke, 1984), gives the amount that the variance is reduced when the random field is averaged over a region of size $W_f \times H$. Note that the dependence of the averaging region on H is apparently only valid for the test case considered; if the footing is founded on a much deeper soil mass, one would not expect to average over the entire depth due to stress distribution with depth. This issue needs additional study.

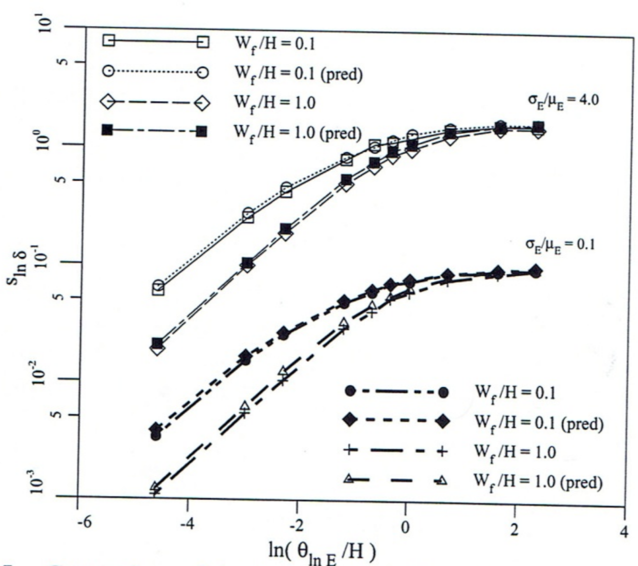

Figure 5. Comparison of simulation estimated standard deviation of log-settlement with theoretical estimate, Eq. (9).

For the isotropic Gauss-Markov correlation function used to represent the $\ln(E)$ random field (Eq. 1), the variance function is closely approximated by

$$\gamma(d_1, d_2;\, \theta) = \tfrac{1}{2}\left[\gamma(d_1)\gamma(d_2|d_1) + \gamma(d_2)\gamma(d_1|d_2)\right] \tag{10}$$

where,

$$\gamma(d_i) = \left[1 + \left(\frac{d_i}{\theta}\right)^{\frac{3}{2}}\right]^{-\frac{2}{3}}, \qquad \gamma(d_i|d_j) = \left[1 + \left(\frac{d_i}{R_j}\right)^{\frac{3}{2}}\right]^{-\frac{2}{3}} \tag{11a}$$

$$R_j = \theta\left[\tfrac{\pi}{2} + \left(1 - \tfrac{\pi}{2}\right)\exp\left\{-\left(\frac{d_j}{\frac{\pi}{2}\theta}\right)^2\right\}\right] \tag{11b}$$

in which d_i are dimensions of the averaging region. Predictions of $\sigma_{\ln\delta}$ using Eq. (9) are plotted in Figure 5 against the simulation results for the largest and smallest footing widths and σ_E/μ_E values considered in this study. The agreement is remarkable. Intermediate cases show similar, if not better agreement with predictions.

Single Footing Example

Consider a single footing of width W_f = 2.0 m to be founded on a soil layer of depth 10.0 m and which will support a load P = 1000 kN. Suppose also that samples taken at a nearby location [1] have allowed the estimation of the elastic modulus mean and standard deviation at the site to be 40 MPa and 40 MPa respectively. Similarly, nearby test results on a regular array have resulted in an estimated scale of fluctuation, $\theta_{\ln E} = 3.0$ m. Assume also that Poisson's ratio is 0.25.

The results from the previous section can be used to estimate the probability that the settlement under the footing will not exceed 0.10 m as follows;

1) A deterministic finite element analysis of the given problem with elastic modulus everywhere equal to μ_E = 40 MPa gives a deterministic settlement of δ_{det} = 0.03531 (note that Eq. 7 gives $\delta_{det} = 0.03604$, a relative difference of 2%).
2) for $W_f/H = 0.2$ and $\theta_{\ln E} = 3$, use Eq. (8) to compute α_2,

$$\alpha_2 = 0.5 + \frac{0.041}{\sqrt{0.2}}\exp\left\{-\tfrac{1}{4}\left(\ln(3/10)+1\right)^2\right\} = 0.5907$$

3) compute variance of log-elastic modulus,

$$\sigma^2_{\ln E} = \ln\left(1 + \left(\frac{\sigma_E}{\mu_E}\right)^2\right) = \ln(2) = 0.69315$$

$$\sigma_{\ln E} = 0.83256$$

4) compute mean of log-settlement,

$$\mu_{\ln\delta} = \ln(\delta_{det}) + \alpha_2\sigma^2_{\ln E} = -3.3437 + 0.5601(0.69315) = -2.9341$$

5) compute standard deviation of log-settlement using Eq.'s (9) through (11),

$$\gamma(W_f) = \left[1 + (W_f/\theta_{\ln E})^{3/2}\right]^{-2/3} = \left[1 + (2/3)^{3/2}\right]^{-2/3} = 0.74847$$

[1] Note that if elastic modulus measurements were taken at the site itself, then the results presented in the previous section would not be applicable: When information about the actual site (beyond statistical information) is known, then the site variability is considerably reduced.

$$\gamma(H) = \left[1 + (H/\theta_{\ln E})^{3/2}\right]^{-2/3} = \left[1 + (10/3)^{3/2}\right]^{-2/3} = 0.27107$$

$$R_1 = 3\left[\tfrac{\pi}{2} + \left(1 - \tfrac{\pi}{2}\right)\exp\left\{-\left(\frac{W_f}{\frac{\pi}{2}\theta_{\ln E}}\right)^2\right\}\right] = 3.28226$$

$$R_2 = 3\left[\tfrac{\pi}{2} + \left(1 - \tfrac{\pi}{2}\right)\exp\left\{-\left(\frac{H}{\frac{\pi}{2}\theta_{\ln E}}\right)^2\right\}\right] = 4.69343$$

$$\gamma(W_f|H) = \left[1 + (W_f/R_2)^{3/2}\right]^{-2/3} = 0.84907$$

$$\gamma(H|W_f) = \left[1 + (H/R_1)^{3/2}\right]^{-2/3} = 0.29261$$

$$\gamma(W_f, H; \theta_{\ln E}) = \tfrac{1}{2}\left[\gamma(W_f)\gamma(H|W_f) + \gamma(H)\gamma(W_f|H)\right] = 0.22458$$

$$\sigma_{\ln\delta} = \sqrt{\gamma(W_f, H; \theta_{\ln E})}\,\sigma_{\ln E} = \sqrt{0.22458}(0.83256) = 0.39455$$

Aside: for $\mu_{\ln\delta} = -2.9341$ and $\sigma_{\ln\delta} = 0.39455$, the corresponding settlement mean and variance can be obtained from the transformations

$$\mu_\delta = \exp\{\mu_{\ln\delta} + \tfrac{1}{2}\sigma^2_{\ln\delta}\} = 0.0575 \quad \text{m}$$

$$\sigma_\delta = \mu_\delta\sqrt{e^{\sigma^2_{\ln\delta}} - 1} = 0.0236 \quad \text{m}$$

A trial run of 2000 realizations for this problem gives $m_\delta = 0.0582$ and $s_\delta = 0.0219$ for relative differences of 1.2% and 7.7% respectively. The estimated standard error on m_δ is approximately 0.0005 for 2000 realizations.

6) compute the desired probability using the lognormal distribution,

$$\begin{aligned} \mathrm{P}\left[\delta \le 0.10\right] &= \Phi\left(\frac{\ln(0.10) - \mu_{\ln\delta}}{\sigma_{\ln\delta}}\right) \\ &= \Phi(1.6006) \\ &= 0.945 \end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution, whose table of values can be found in any good probability textbook.

The simulation run for this problem yielded 1892 samples out of 2000 having settlement less than 0.10 m. This gives a simulation based estimate of the above probability of 0.946, a relative difference of only 0.1%. Although this is very good accuracy, one must a bit cautious since if the probability in question had been $\mathrm{P}\left[\delta > 0.10\right]$ then the relative error becomes 1.9%. It is expected that probabilities estimated farther out in the tail of the distribution may have even larger differences with simulation results unless the simulation is carried out over very many more realizations, and this is yet to be verified.

TWO FOOTING CASE

Having established with reasonable confidence the distribution associated with settlement under a single footing founded on a soil layer, attention can now be turned to the more difficult problem of finding a suitable distribution to model differential settlement between footings. Analytically, if δ_1 is the settlement under the left footing shown in Figure 1 and δ_2 is the settlement of the right footing, then according to the results of the previous section, δ_1 and δ_2 are joint lognormally distributed random variables following the bivariate distribution

$$f_{\delta_1 \delta_2}(x, y) = \frac{1}{2\pi\sigma^2_{\ln\delta} rxy} \exp\left\{ -\frac{1}{2r^2}\left[\Psi_x^2 - \rho_{\ln\delta}\Psi_x\Psi_y + \Psi_y^2\right]\right\} \tag{12}$$

for $x \geq 0$, $y \geq 0$, where $\Psi_x = (\ln x - \mu_{\ln\delta})/\sigma_{\ln\delta}$, $\Psi_y = (\ln x - \mu_{\ln\delta})/\sigma_{\ln\delta}$, and where $r^2 = 1 - \rho^2_{\ln\delta}$ with $\rho_{\ln\delta}$ being the correlation coefficient between the log-settlement of the two footings. It is assumed in the above that δ_1 and δ_2 have the same mean and variance, which, for the symmetric conditions shown in Figure 1(b), is a reasonable assumption.

If the differential settlement between footings is defined by

$$\Delta = |\delta_1 - \delta_2| \tag{13}$$

then the distribution of Δ is given by

$$f_\Delta(x) = 2\int_0^\infty f_{\delta_1\delta_2}(x + y, y)\, dy \tag{14}$$

Unfortunately, this integral cannot be solved analytically insofar as the authors are aware, although for design purposes it can be estimated using any available reliability tool, such as first- or second-order reliability methods. Such numerical approximations to Eq. (14) are being investigated for a future publication. It is not hard to show that the variance of Δ can be written,

$$\sigma^2_\Delta = 2(1 - \rho_\delta)\sigma^2_\delta - \mu^2_\Delta \tag{15}$$

where ρ_δ is the correlation coefficient between δ_1 and δ_2.

Figure 6 shows a typical histogram of differential settlement between the two equal sized footings. Superimposed on the histogram is a trial exponential distribution having the form

$$f_\Delta(x) = \frac{1}{\mu_\Delta}\exp\{-x/\mu_\Delta\} \tag{16}$$

with μ_Δ taken as 0.8998 which is the data average. Although this distribution fails the Chi-Square goodness-of-fit test, it appears to capture the major trends in the histogram, particularly in the tail.

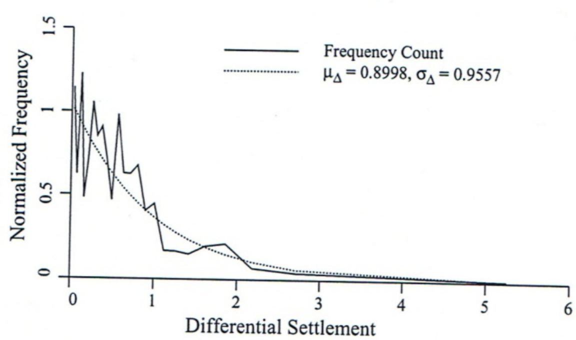

Figure 6. Frequency histogram and fitted distribution for differential settlement under two equal sized footings.

Figure 7 shows the estimated mean and standard deviation of Δ as functions of $\theta_{\ln E}/H$, σ_E/μ_E, for $W_f/H = 0.5$. The other two footing widths considered (0.1 and 0.3) are of similar form. The two plots are nearly identical, suggesting that perhaps it is not unreasonable to take a one-parameter exponential distribution as representative of the differential settlement (for which $\sigma_\Delta = \mu_\Delta$). A trial function of the form

$$\mu_\Delta^2 = \beta(1-\rho_\delta)\sigma_\delta^2 \tag{17}$$

will be investigated (see Eq. 15) to predict the parameter of the exponential distribution, for some positive constant β. Note that when $\theta_{\ln E} \to 0$, the previous section predicted $\sigma_\delta^2 \to 0$, and when $\theta_{\ln E} \to \infty$, the correlation coefficient between the footing settlements, $\rho_\delta \to 1$. Thus Eq. (17) is in agreement with the observation that differential settlements are expected to go to zero for both very small and very large values of $\theta_{\ln E}$.

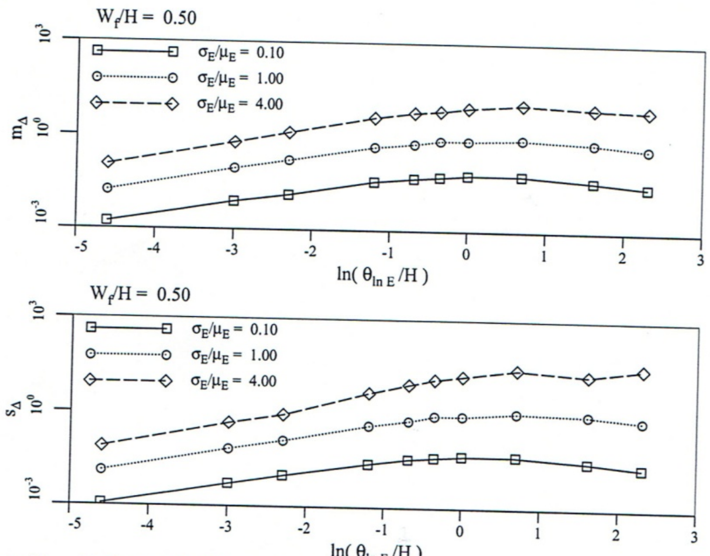

Figure 7. Estimated differential settlement mean and standard deviation.

The factor ρ_δ in Eq. (17) can be obtained by first considering the correlation, $\rho_{\ln\delta}$, between the two local averages of the $\ln(E)$ field under the two footings; these local averages are of dimension W_f in width by H in height and are separated, center to center, by the distance D. The correlation, $\rho_{\ln\delta}$, can be found using the variance function as

$$\rho_{\ln\delta} = \frac{1}{2W_f^2}\left\{(D-W_f)^2\gamma(D-W_f, H)+(D+W_f)^2\gamma(D+W_f, H)-2D^2\gamma(D, H)\right\} \tag{18}$$

where the $(;\theta_{\ln E})$ notation is now dropped for convenience, it being understood that the variance function is dependent on $\theta_{\ln E}$. Calculating $\mu_{\ln\delta}$ and $\sigma_{\ln\delta}$ as in the previous section (Eq's 6 to 9) then allows the computation of μ_δ and σ_δ^2;

$$\mu_\delta = \exp\{\mu_{\ln\delta} + \tfrac{1}{2}\sigma_{\ln\delta}^2\} \tag{19a}$$

$$\sigma_\delta^2 = \mu_\delta^2(e^{\sigma_{\ln\delta}^2} - 1) \tag{19b}$$

With these results, the correlation ρ_δ can be found from

$$\rho_\delta = \frac{\exp\{\rho_{\ln\delta}\sigma^2_{\ln\delta}\} - 1}{\exp\{\sigma^2_{\ln\delta}\} - 1} \tag{20}$$

and μ_Δ can now be estimated using Eq. (17) if a suitable value of β is found.

To test the ability of the assumed distribution to accurately estimate probabilities, the probability

$$\mathrm{P}\left[\Delta \le \alpha\mu_\Delta\right] = 1 - e^{\alpha} \tag{21}$$

for α varying from 0.5 to 4.0, is plotted against the corresponding probabilities estimated directly from the simulation results. After some trial and error, a β value of $2/3$ was found to give the most accurate probabilities on average over the range of W_f, σ_E/μ_E, and $\theta_{\ln E}$ considered here. Figure 8 shows the predicted probabilities using Eq. (17) with $\beta = 2/3$ alongside the estimated probabilities averaged over all σ_E/μ_E and $\theta_{\ln E}$ parameter values. When results are not averaged over parameter values, the agreement is typically less good away from the tails (some above, some below) but generally reasonable in the tails.

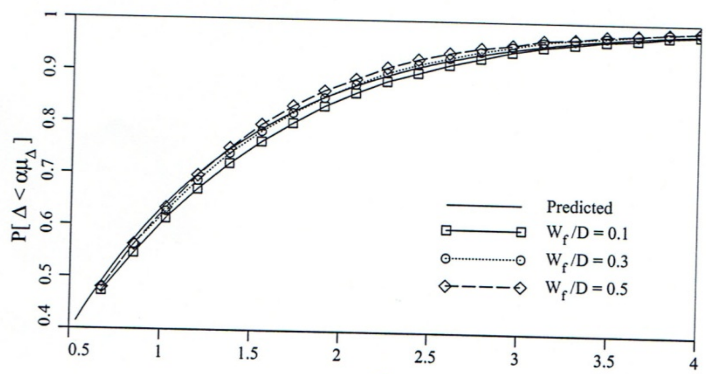

Figure 8. Simulation based estimates of $\mathrm{P}\left[\Delta \le \alpha\mu_\Delta\right]$, averaged over all $\theta_{\ln E}$ and σ_E/μ_E cases, compared to that predicted by Eq.'s (17) and (21).

CONCLUSIONS

On the basis of this initial simulation study, which is by no means complete, some tentative observations are made as follows.

It appears that the settlement under a footing founded on a spatially random elastic modulus field of finite depth overlying bedrock is reasonably well represented by a lognormal distribution, if E is also lognormally distributed, with parameters $\mu_{\ln\delta}$ and $\sigma^2_{\ln\delta}$. The first parameter, $\mu_{\ln\delta}$, is dependent on the mean and variance of the underlying elastic modulus field and may be largely derived by considering limiting values of $\theta_{\ln E}$. Although there is some question as to why the slope correction term appearing in Eq. (8) is necessary, including it yields quite accurate estimates of the mean log-settlement. It is significant to note that the second parameter, $\sigma^2_{\ln\delta}$, is very well approximated by the variance of a local average of the elastic modulus field in the region directly under the footing. This gives the prediction of $\sigma^2_{\ln\delta}$ some generality that could possibly extend beyond the actual range of simulation results considered herein if a suitable averaging domain can be defined. Once the statistics of the settlement, $\mu_{\ln\delta}$ and $\sigma^2_{\ln\delta}$, have been computed using Eq.'s 6 to 9 the estimation of probabilities associated with settlement involves little more than referring to a standard normal distribution table.

The differential settlement follows a more complicated distribution than that of settlement itself (see Eq. 14). This is seen also in the differential settlement histograms which tend to be quite erratic with long tails. Clearly the difference between two lognormally distributed random variables is not exponentially distributed. Neither does this difference follow a normal distribution, the normal distribution falling off far too rapidly in the tails. For an accurate estimation of probability relating to differential settlement where it can be assumed that footing settlement is lognormally distributed, Eq. (14) should be numerically integrated. This approach was not pursued in this study since it was observed that the tail of the differential settlement histogram was reasonably well approximated by an exponential distribution. The results suggest a relatively simple approach to obtaining 'ball-park' estimates of probabilities associated with differential settlement that involves, again, statistics of the underlying elastic modulus field and local averages of the field directly under the footings. In that the statistical parameters of the underlying elastic modulus field are themselves estimates, that may or may not be very accurate, the proposed probability predictions may be reasonable.

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